An Improved Fully Dynamic Algorithm for Counting 4-Cycles in General Graphs

Vihan Shah

Computer Science
University of Waterloo
vihan.shah@uwaterloo.ca

Abstract

We study subgraph counting over fully dynamic graphs, which undergo edge insertions and deletions. Counting subgraphs is a fundamental problem in graph theory with numerous applications across various fields, including database theory, social network analysis, and computational biology. In database theory, we can use dynamic subgraph counting algorithms on layered graphs to maintain the sizes of joins of databases that undergo updates. For instance let R, S, and T be relations that have schemas (A, B), (B, C), and (C, A) respectively. Then the size of the join of R with S with T is given by the number of triangles in the corresponding layered graph where there is a layer for each attribute, the vertices are the attribute values and the edges represent the tuples of attribute values in the relations.

Maintaining the number of triangles in fully dynamic graphs is very well studied and has an upper bound of $O(\sqrt{m})$ for the update time [KNN+20]. There is also a conditional lower bound of $\Omega(m^{1/2-\varepsilon})$ for any constant $\varepsilon > 0$, for the update time [HKNS15] under the Online Matrix-Vector (OMv) conjecture implying that $O(\sqrt{m})$ is the "right answer" for the update time of counting triangles.

More recently, [HHH22] studied the problem of maintaining the number of 4-cycles in fully dynamic graphs and designed an algorithm with $O(m^{2/3})$ update time which is a natural generalization of the approach for counting triangles. They also studied the problem of counting 4-cliques showing that the folklore upper bound of $O(m)$ for the update time is tight under the static combinatorial 4-clique conjecture by giving a lower bound of $\Omega(m^{1-\varepsilon})$ for any $\varepsilon > 0$. Thus, it seems natural that $O(m^{2/3})$ might be the correct answer for the complexity of the update time for counting 4-cycles.

In this work, we present an improved algorithm for maintaining the number of 4-cycles in fully dynamic graphs. Our algorithm achieves a worst-case update time of $O(m^{2/3-\varepsilon})$ for some constant $\varepsilon > 0$. We also show that the problem of counting 4-cycles is equivalent in layered graphs and general graphs. Our approach crucially uses fast matrix multiplication and leverages recent developments therein to get an improved runtime. Using the current best value of the matrix multiplication exponent $\omega = 2.371339$ we get $\varepsilon = 0.009811$ and if we assume the best possible exponent i.e. $\omega = 2$ then we get $\varepsilon = 1/24$. There is also a lower bound of $\Omega(m^{1/2-\varepsilon})$ for any constant $\varepsilon > 0$, for the update time [HKNS15, HHH22], so there is still a big gap between the best known upper and lower bounds. The key message of our paper is demonstrating that $O(m^{2/3})$ is *not* the correct answer for the complexity of the update time.

References

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